Representation of the quantum algebra $SU_q(2)$ in the basis with diagonal "J_x" generator

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Abstract

Generators of the quantum $SU_q(2)$ algebra are obtained in the explicit form in the basis where the operator $\exp \frac{J_z}{2} J_x \exp \frac{J_z}{2}$ is diagonal. It is shown that the solution of this problem is related to the representation theory of the two-dimensional algebra $[s, r] = \tanh t\,(s^2 - r^2 + 1)$. The relevance of such basis to some problems of quantum optics is discussed.

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1 Introduction

The common realizations of the quantum algebra $\mathrm{SU}_q(2)$ are connected with the basis where the generator J_z is diagonal. This tradition can be explained by the fact that this algebra is invariant under the U(1)-rotations in x, y plane. However, in the recent paper [1] it was shown that in each irreducible representation of $\mathrm{SU}_q(2)$ algebra the operator $\exp\frac{J_z t}{2} J_x \exp\frac{J_z t}{2}$ can be diagonalized, and its spectrum does not depend on the choice of the representation.

The aim of the present paper is to show that this result is not occasional but has deep foundations. We will be able to construct in the explicit form generators of irreducible representations of $SU_q(2)$ algebra in the above mentioned basis. The strategy of our calculations is based upon the quasi-classical nature of representation of the quantum algebras, as it was observed in [2]. Firstly, we will find the representation of the corresponding functional quantum group and then generalize these results properly to the quantum algebra case.

The paper is organized in the following way. In Section 2 we consider the basis in quantum algebra $SU_q(2)$, where operator "J_x" is diagonal. We show that the $SU_q(2)$ generators can be written in terms of only two operators from the enveloping algebra (see S and r below, s is proportional to "J_x" generator), which obey a single commutation relations. In Sections 3 and 4 we find realization of this new $\{s, r\}$ algebrain terms of generators of Heisenberg-Weyl algebra. We start with the consideration at the classical level, when all commutators are changed for the Poisson brackets (a functional group case) in Section 3. In Section 4 we give solution for the "quantum" case. The limit, when deformation parameter $t \to 0, (q \to 1)$ is considered in Section 5. The raising and lowering operators in the new basis are found in Sectoion 6. Nontrivial role of the Casimir operator is discussed in Section 7, where finite-dimensional representations of $\{s, r\}$ algebra are represented in explicit form. Conclusions are given in Section 8.

2 New form of commutation relations of the $\mathrm{SU}_q(2)$ algebra

The common form of the commutation relations for the generators $J^\pm$, H of the $\mathrm{SU}_q(2)$ algebra is :

$$[H, J^\pm] = \pm 2J^\pm, \quad [J^+, J^-] = \frac{\sinh(tH)}{\sinh t}, \quad J_x = J^+ + J^-, \quad H = 2J_z, \tag{2.1}$$

where $t = \log q$ is the deformation parameter. The Casimir operator that commutes with all generators of the algebra has the form:

$$\mathcal{C} = J^+J^- + J^-J^+ + \frac{\cosh t}{\sinh^2 t}\,(\cosh(tH) - 1)$$

Let us introduce the generators

$$T^\pm = \exp\frac{Ht}{4}\, J^\pm \exp\frac{Ht}{4}, \qquad R = \exp(tH)\,.$$

The commutation relations determining the $\mathrm{SU}_q(2)$ algebra (2.1) may be rewritten as follows:

$$RT^\pm = e^{\pm 2t}\, T^\pm R, \qquad e^t\, T^+T^- - e^{-t}\, T^-T^+ = \frac{R^2 - 1}{2\sinh t}\,. \tag{2.2}$$

The Casimir operator takes the form:

$$\mathcal{C} = R^{-1}\left(e^{t}\,T^{+}T^{-} + e^{-t}\,T^{-}T^{+} + \frac{(R-1)^2\cosh t}{2\sinh^2 t}\right)$$

The remarkable property of the commutation relations (2.2) is that the last equation may be rewritten in terms of only two (not three!) generators of the initial algebra. Introducing the pair of operators

$$Q^{\pm} = T^{\pm} \pm \frac{R}{2\sinh t},$$

we rewrite the last equation of (2.2) in the form:

$$e^{t}\,Q^{+}Q^{-} - e^{-t}\,Q^{-}Q^{+} = -\frac{1}{2\sinh t}\,. \tag{2.3}$$

Note that

$$Q^{+} + Q^{-} = T^{+} + T^{-} = \exp\frac{J_z t}{2} J_x \exp\frac{J_z t}{2}, \quad H = 2J_z$$

is exactly the operator which was diagonalized in [1].

Now we want to consider the representations of $\mathrm{SU}_q(2)$ algebra in the basis where this operator is diagonal. Our strategy is as follows: We will try to find solutions of the single equation (2.3), and express R in terms of Q^{+}, Q^{-} and the Casimir operator. We start with the less difficult second part of the problem.

Let us introduce the generators

$$s = (Q^{+} + Q^{-})\sinh t, \qquad r = (Q^{+} - Q^{-})\sinh t\,.$$

We now can rewrite (2.3) in the more attractive form:

$$[s, r] = (s^2 - r^2 + 1)\tanh t\,. \tag{2.4}$$

It is clear that the two-dimensional quantum algebra (2.3) is invariant with respect to $U(1)$ transformations $Q^{\pm} \to \exp \pm\alpha Q^{\pm}$. As a consequence, the algebra (2.4) is invariant with respect to the Lorenz transformations in the two-dimensional r, s plane.

By means of purely algebraic manipulations using the above formulae, it is not difficult to rewrite the Casimir operator in terms of s, r, R operators:

$$\mathcal{C} = \sinh^{-2} t R^{-1}\left[\frac{s^2 - r^2 + 1}{2\cosh t} + R\left((r-1)\,\cosh t - s\,\sinh t\right)\right]$$

The remarkable fact is that the quadratic terms (R^2) in the square brackets in the right-hand side of the last expression are canceled and this give the possibility to express R in terms of s, r operators under the fixed numerical value of the Casimir operator,

$$R = K^{-1}\,(s^2 - r^2 + 1)/2\,, \quad K = \cosh t\,\left(\mathcal{C}\sinh^2 t + (1-r)\,\cosh t - s\,\sinh t\right)\,, \tag{2.5}$$

provided that the inverse of the operator K exists. We recall that the operators s, r do not commute and thus the order of the factors in the above formulae is important.

In conclusion of this section we describe the way of proving that the operator R as defined by (2.5), indeed satisfies the first pair of equations of initial quantum algebra (2.2). We restrict ourselves by + sign in Eq. (2.2).

In accordance with the above definitions,

$$T^{\pm} = \frac{s \pm r \mp R}{2 \sinh t} .$$

Thus, for the checking of (2.2) it is necessary to move $s + r$ throughout R. The following formulae explain this procedure:

$$(s - r)(s + r) + 1 = s^2 - r^2 + [s, r] + 1 = \frac{\exp t}{\cosh t} (s^2 - r^2 + 1)$$

$$(s + r)(s - r) + 1 = s^2 - r^2 - [s, r] + 1 = \frac{\exp(-t)}{\cosh t} (s^2 - r^2 + 1)$$

$$(s^2 - r^2 + 1)(s + r) = e^t \cosh t\, ((s + r)(s - r) + 1)(s + r) =$$

$$e^t \cosh t\, (s + r)((s - r)(s + r) + 1) = e^{2t} (s + r)(s^2 - r^2 + 1) .$$

These formulae explain the action connected with the second factor of $R = K^{-1}(s^2 - r^2 + 1)/2$. The calculations connected with the first factor are based upon the commutator:

$$[K,\ s + r] = -4e^{-t} \cosh t\, [s, r] = -2(1 - e^{-2t})(s^2 - r^2 + 1) .$$

Multiplying the last equations from the left and from the right by K^{-1} we come to the formula

$$[s + r, K^{-1}] = -2(1 - e^{-2t})\, K^{-1}(s^2 - r^2 + 1) K^{-1}$$

which completes checking of Eq. (2.2).

3 Resolution of the two-dimensional algebra on the level of the functional group

On the functional group level [3] commutator in (2.4) is replaced by Poisson brackets. We thus have,

$$\{s, r\} = \tanh t\, (s^2 - r^2 + 1) . \tag{3.6}$$

In accordance with the Darboux's theorem [3], s, r from (3.6) may be expressed as functions of the pair of canonically conjugated variables p, x, $\{p, x\} = 1$.

To keep the correspondence with the results [1] let us choose $s = \sinh tp$ and $r = r(x, p)$. Eq. (3.6) takes on the form

$$t \cosh(tp)\, \frac{dr}{dx} = \tanh t\, (\cosh^2 tp - r^2)$$

with the obvious general solution:

$$r = \cosh(tp)\, \coth\left(\nu x + \phi(p)\right) , \qquad \nu \equiv \frac{\tanh t}{t} .$$

4 The case of the quantum algebra

Reminding about the quasi-classical nature of the problem of construction of irreducible representation of the quantum algebras [2], and keeping in mind the results of the consideration on the functional group level (the previous section), let us try to resolve the quantum algebra (2.4) with the help of the substitution:

$$s = \sinh(tp)\,, \qquad r = \frac{A(x)(x)\,e^{tp} + e^{-tp}B(x)}{2}\,, \tag{4.7}$$

where now p, x are the generators of the Heisenberg-Weil algebra, $[p, x] = 1$. (Note that the term $e^{-tp}B(x)$ in the expression for r is understood as a multiplication of two operators but not as an action of the first operator on the function $B(x)$.) We use below the following obvious relation from the theory of Heisenberg-Weil algebra:

$$e^{\mu p}F(x) = F(x+\mu)e^{\mu p}\,.$$

Substituting (4.7) into (2.4) and equating to zero the coefficients at the operators $\exp 2tp, \exp -2tp, I$ we come to the following system of equations for unknown functions $A(x), B(x)$:

$$A(x+t) - A(x) = \tanh t\,(1 - A(x+t)A(x)), \quad B(x+t) - B(x) = \tanh t\,(1 - B(x+t)B(x))$$

$$A(x) - A(x-t) + B(x) - B(x-t) = \tanh t\,(2 - A(x)B(x) - A(x-t)B(x-t)) \tag{4.8}$$

Substituting the first and the second equations (with the arguments shifted by $-t$) into the the third one of (4.8) we obtain the selfconsistency condition of (4.8):

$$(A(x) - B(x-t))(B(x) - A(x-t)) = 0\,. \tag{4.9}$$

Choosing the first possibility we have $B(x) = A(x-t)$, and the system (4.8) becomes equivalent to a a single equation:

$$A(x+t) - A(x) = \tanh t\,(1 - A(x+t)A(x))$$

with the nontrivial solution:

$$A(x) = \coth(x+F)\,.$$

We finally obtain the explicit realization of the generators of the two-dimensional quantum group (2.4) in terms of the generators of the Heisenberg algebra $(\hat{p}, \hat{x})$:

$$s = \sinh t\hat{p}, \quad r = \cosh t\hat{p} \times \coth(\hat{x}+F), \quad or \quad r = \coth(\hat{x}+F) \times \cosh t\hat{p} \tag{4.10}$$

where $F \equiv F(t, \hat{p})$ and the two forms of the operator r corresponds to the two possible ways of resolution the selfconsistency conditions (4.9).

5 The limit $t \to 0$

To get an experience of working in the above basis let us consider firstly the limit case $t \to 0$, assuming that the quantum algebra is transformed into the usual $SU(2)$ algebra in this limit:

$$s \to ts_1, \quad r \to 1 + tr_1 + t^2 r_2, \quad t \to 0 \tag{5.11}$$

(As will be clear below, it is not necessary to take into account terms $\sim t^2$ in s).

Under such an assumption the principal equation (2.4) (in the leading order with respect to t) takes on the form:

$$[s_1, r_1] = -2r_1 \,. \tag{5.12}$$

On the other hand, from the definitions we have,

$$\begin{aligned} r &= \sinh t\,(T^+ - T^-) + R \to 1 + t(H + J^+ - J^-), \\ s &= \sinh t(T^+ + T^-) \to t(J^+ + J^-), \qquad R \to 1 + tH \end{aligned}$$

and indeed, Eq. (5.12) is consistent with the last relations:

$$[J^+ + J^-, H + J^+ - J^-] = -2(H + J^+ - J^-)$$

Now let us calculate H from (2.5). We obtain in the limit $t \to 0$,

$$r_1 R \to r_1 + t\left(\frac{1}{2}(r_1^2 - s_1^2) - s_1 + \mathcal{C}\right) .$$

We thus recover the following realization of A_1 algebra in the "L_x" basis:

$$s_1 = J^+ + J^-, \quad r_1 = H + J^+ - J^-, \quad r_1 H = \frac{1}{2}(r_1^2 - s_1^2) - s_1 + \mathcal{C} \,. \tag{5.13}$$

6 Rising and lowering operators in the "J_x" basis

If we assume that in the "L_x" basis the operator s is diagonal, then, in correspondence with the result of [1], its spectrum is:

$$\hat{s}|M\rangle = \sinh(tM)|M\rangle.$$

The fact of possible existence of rising (lowering) operators $\Theta^\pm$ may be expressed by the equations:

$$\sinh(t\hat{p})\,\Theta^\pm = \Theta^\pm \sinh(t(\hat{p} + 2)) \,. \tag{6.14}$$

It is also assumed that the generators $\Theta^\pm$ can be constructed from the three generators of quantum algebra $Q^\pm, R$ or $J^\pm, H$.

The direct solution of this problem is unknown for us and we will try to solve it by the trick similar to the one used in resolution of the two-dimensional quantum group (4.7) (see section 4). We assume that the following substitution for generators $\Theta^\pm$ works:

$$\Theta^\pm = F^\pm(\hat{p})\,(A^\pm(x)\exp t\hat{p} + \exp(-t\hat{p})B^\pm(x)) \tag{6.15}$$

Replacing the last expression into (6.14) we come to the result:

$$\Theta^\pm = F^\pm(\hat{p})\exp(\pm 2x + f^\pm(\hat{p})) \tag{6.16}$$

The question remains, how these operators can be expressed in terms of quantum algebra generators under realization of section 4.

7 Matrix realization of the two-dimensional algebra

As it follows from Section 5, in the limit $t \to 0$ r_1 is the lower triangular matrix if s_1 is the diagonal one. It is surprising that the principal equation (2.4) can be also resolved on the level of $(2l+1) \times (2l+1)$-matrices (l being integer or half-integer positive number). Then s is a diagonal matrix with matrix elements $s_{i,j} = \delta_{i,j} \sinh[t(2l+2-2j)],\ (1 \leq i, j \leq 2l+1)$ and r the matrix with the matrix elements $r_{i,j}$ different from zero on the principal diagonal and below $(i \geq j)$. The diagonal elements of r are determined uniquely from (2.4) with the result:

$$r_{j,j} = \pm \cosh[t(2l+2-2j)]\,.$$

If we want to have a correct limit $t \to 0$, in correspondence with the results of the section 5, the positive signs have to be taken.

Let us now rewrite (2.4) in terms of the matrix elements (i (j) is the number of the row (column) $j \leq i$):

$$[\sinh(t(2l+2-2i)) - \sinh(t(2l+2-2j))]\alpha_{ij}$$
$$= -\tanh t \left\{ [\cosh(t(2l+2-2i)) + \cosh(t(2l+2-2j))]\, \alpha_{ij} + \sum_{k=j+1}^{i-1} \alpha_{ik}\alpha_{kj} \right\}.$$

The last equation may be rewritten in more compact form:

$$2\cosh[t(2l+2-i-j)]\ \sinh[t(i-j-1)]\, \alpha_{ij} = \sinh t \sum_{k=j+1}^{i-1} \alpha_{ik}\alpha_{kj}\,. \tag{7.17}$$

From (7.17) it follows immediately that $\alpha_{i,i-1} \equiv \alpha_i$ are arbitrary c-numbers. All other matrix elements of the lower triangular matrix α_{ij} may be presented in the form:

$$\alpha_{ij} = c_{ij}(t)\alpha_i\alpha_{i-1}.....\alpha_j$$

where the functions $c_{ij}(t)$ are easily found by induction. Firstly, let $j = i-2$. In this case (7.17 is equivalent to

$$2\cosh[t(2l+4-i)]\alpha_{ii-2} = \alpha_{i,i-1}\alpha_{i-1,i-2} \equiv \alpha_i\alpha_{i-1}$$

or

$$c_{i,i-2} = \frac{1}{2\cosh[t(2l+4-2i)]}$$

The next step $j = i-3$ leads to:

$$c_{i,i-3} = \frac{1}{2\cosh t(2l+4-2i)}\,\frac{1}{2\cosh t(2l+6-2i)}$$

It is not difficult to prove by induction that

$$c_{i,i-s} = \frac{1}{2^{s-1}} \prod_{k=2}^{s} \cosh^{-1}[t(2l+2k-2i)]\,. \tag{7.18}$$

and

$$\alpha_{i,i-s} = c_{i,i-s} \prod_{k=1}^{s} \alpha_{i+1-k}$$

We thus have found (but not proved its uniqueness) realization of the two-dimensional algebra (2.4) by $(2l+1)\times(2l+1)$- matrices depending on $2l$ arbitrary parameters α_i.

In fact the above construction is invariant under similarity transformations by the diagonal matrices. With the help of such a transformation all different from zero parameters α_i may be made equal to unity and we will have no arbitrary parameters in the above construction. However, for further consideration it will be convenient do not fix these parameters.

The reader may ask, with sarcasm, what is the role of Casimir operator in the whole construction? The answer is not trivial and unexpected. Up to now, we have constructed the representation of (2.4) algebra only, and after adding to it the operator R from (2.5) the representation of (2.2) algebra, but not the representation of the initial $SU_q(2)$ one, Eq. (2.1). If we want to come back to the algebra (2.1) it is necessary to choose from all representations of (2.2) algebra only those for which operator R is invertible and $R^{-1}(t) = \exp -tH$ in agreement with its definition. However, $\det R$ defining by (2.5) is equal exactly to 0 under the matrix realization of this section. Indeed matrix $(s^2 - r^2 + 1)$ is just lower triangular matrix with zeros on its main diagonal. We thus need to come back to the equation (2.5) in its initial form:

$$RK = \frac{1}{2}\left(s^2 - r^2 + 1\right), \quad K = \cosh t\,\left(\mathcal{C}\sinh^2 t + (1-r)\cosh t - s\sinh t\right). \tag{7.19}$$

If we want to have a realization with $\det R$ different from zero, then it follows immediately from (7.19 that $\det K = 0$. K is, in general case (with an arbitrary $\mathcal{C}$ in it), the lower triangular matrix with different from zero diagonal elements. $\det K = 0$ only if one of its diagonal elements is equal to zero. This defines the possible value of $\mathcal{C}$ under which the transition from (2.2) to (2.1) is possible in general. In addition, in each of such cases it is necessary to check whether the condition $R^{-1}(t) = R(-t)$ is satisfied. Below we consider the simplest examples, explaining the situation.

7.1 The case $2l+1=2$

For this particular case, we have:

$$s = \begin{pmatrix} \sinh t & 0 \\ 0 & -\sinh t \end{pmatrix}, \quad r = \begin{pmatrix} \cosh t & 0 \\ \alpha_1 & \cosh t \end{pmatrix}.$$

Substituting these expressions into definition of K matrix (2.5) we obtain:

$$K = \begin{pmatrix} \mathcal{C}\cosh t\sinh^2 t + \cosh^2 t - \cosh t & 0 \\ -\alpha_1\cosh^2 t & \mathcal{C}\cosh t\sinh^2 t + \cosh^2 t - \cosh t\cosh 2t \end{pmatrix}$$

$$\frac{s^2 - r^2 + 1}{2} = \begin{pmatrix} 0 & 0 \\ -\alpha_1\cosh t & 0 \end{pmatrix}$$

Two possibilities arise: $\mathcal{C}\sinh^2 t = 1 - \cosh t$ and in this case equation (7.19) has no solution with $\det R \neq 0$. In the second case, $\mathcal{C}\sinh^2 t = 2\sinh\frac{3}{2}t\sinh\frac{1}{2}t$ and the equation for determining

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

takes the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cosh t(\cosh 2t - 1) & 0 \\ -\alpha_1\cosh^2 t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\alpha_1\cosh t & 0 \end{pmatrix}$$

with the general solution:

$$R = \begin{pmatrix} A\alpha_1 \cosh t & A(\cosh 2t - 1) \\ B\alpha_1 \cosh t & \cosh^{-1} t + B(\cosh 2t - 1) \end{pmatrix}, \quad \det R = A\alpha_1$$

where A, B arbitrary constants.

Choosing

$$A\alpha_1 = 1, \quad A = (2\sinh t)^{-1}, \quad B = (2\cosh t)^{-1}, \quad \alpha_1 = 2\sinh t$$

we obtain R in the form:

$$R = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

which satisfies all of the conditions for the operator R.

7.2 The case $2l + 1 = 3$

$$s = \begin{pmatrix} \sinh 2t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sinh 2t \end{pmatrix}, \quad r = \begin{pmatrix} \cosh 2t & 0 & 0 \\ \alpha_1 & 1 & 0 \\ \frac{\alpha_1\alpha_2}{2} & \alpha_2 & \cosh 2t \end{pmatrix}$$

$$K = \begin{pmatrix} \cosh t(\cosh 3t - \cosh t) & 0 & 0 \\ -\alpha_1 \cosh^2 t & \cosh t(\cosh 3t - \cosh t) & 0 \\ -\frac{\alpha_1\alpha_2}{2} \cosh^2 t & -\alpha_2 \cosh^2 t & 0 \end{pmatrix}$$

$$\frac{s^2 - r^2 + 1}{2} = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha_1 \cosh^2 t & 0 & 0 \\ -\alpha_1\alpha_2 \cosh^2 t & -\alpha_2 \cosh^2 t & 0 \end{pmatrix}$$

the general solution of the equation (7.19) in this case has the form:

$$R = \begin{pmatrix} \frac{T_1\alpha_1\alpha_2}{d}(1 - \frac{d}{2}) & -T_1\alpha_2 & T_1 d \\ \frac{\alpha_1}{d} + \frac{T_2\alpha_1\alpha_2}{d}(1 - \frac{d}{2}) & -T_2\alpha_2 & T_2 d \\ \frac{\alpha_1\alpha_2}{2d} + \frac{T_3\alpha_1\alpha_2}{d}(1 - \frac{d}{2}) & -T_3\alpha_2 & 1 + T_3 d \end{pmatrix}$$

where T_i three arbitrary c-number constants and $(1 - \frac{d}{2}) = \cosh 2t$.

We know that in the case of A_q^1 specter of this operator is $1, \exp \pm 2t$. This means that $DetR = 1, TrR = 2\cosh 2t + 1, TrR^2 = 2\cosh 4t + 1$. This conditions allow to determine all constants T_i:

$$\frac{T_1\alpha_1\alpha_2}{d} = 1, \quad T_2\alpha_2 = 1, \quad T_3 d = 1 + \cosh 2t.$$

Putting $\alpha_1 = \alpha_2 = d$ we evaluate R to the form ($x = 1 - \frac{d}{2}$:

$$R = \begin{pmatrix} x & -1 & 1 \\ 1 + x & -1 & 1 \\ 1 + x^2 & -(1 + x) & (2 + x) \end{pmatrix}$$

The canonical transformation evaluating the last matrix to diagonal form is exactly overgo from the basis with diagonal operator s to the basis where diagonal is R operator. Explicit form of this transformation was found in [1].

8 Conclusions

The rezults of the present paper may be devided into two part: pure mathematical ones and their physical applications. As for the first ones, the principal point is that we are able to reduce the three-dimensional $su_q(2)$ algebra to two-dimensional algebra s, r (2.5). Is it an exidental fact or not? The answer on this question will be possible only if the generalization of the present construction to the case an arbitrary quantum algebra will be found. We hope to take part in the investigation of this question in further publications.

No less interesting point is physical applications of this construction. We would like to say a few words about its relation to a quantum optics problems. Typical optical hamiltonians describing the processas of thre and four-wave mixing, the second and the third-harmonics generation, the interuction of atomic systems with quantized cavity radiation field can be presented in the block diagonal form. Each block is finite dimensional subspace where the Hamiltonian acts as three- diagonal matrix (usually, of very high dimension) whose matrix elements depend on the values of the corresponding integral of motions. The complete solution of the quantummachanical problem would include the diagonalization of all these Hamiltonians.

We hope that the spectrum of "J_x" generators (which in its turn can be considered as some exact or appropriate Hamiltonian for some optical model) found in present paper for arbitrary representation of quantum $su_q(2)$ algebra will be usefull for analitical treatment of quantum optical processes mentioned above.

Acknowledgements

Author would like to thanks A.V.Turbiner,V.Tolstoy and Yu.F.Smirnov for interesting and fruitful discussions.

This work was done under partial support of the Russian Foundation for Fundamental Recearches (RFFI) GRANT-N 98-01-00330.

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